Eur. Phys. J. C 48, 633–639 (2006)
DOI 10.1140/epjc/s10052-006-0024-6

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article – Theoretical Physics

Jordanian twist quantization of $D=4$ Lorentz and Poincaré algebras and $D=3$ contraction limit

A. Borowiec[1], J. Lukierski[1,a], V.N. Tolstoy[1,2]

[1] Institute for Theoretical Physics, University of Wrocław, pl. Maxa Borna 9, 50–205 Wrocław, Poland
[2] Institute of Nuclear Physics, Moscow State University, 119992 Moscow, Russia

Received: 24 April 2006 /
Published online: 24 October 2006 –

Abstract. We describe in detail the two-parameter nonstandard quantum deformation of the $D=4$ Lorentz algebra $\mathfrak{o}(3,1)$, linked with a Jordanian deformation of $\mathfrak{sl}(2;\mathbb{C})$. Using the twist quantization technique we obtain the explicit formulae for the deformed co-products and antipodes. Further extending the considered deformation to the $D=4$ Poincaré algebra we obtain a new Hopf-algebraic deformation of four-dimensional relativistic symmetries with a dimensionless deformation parameter. Finally, we interpret $\mathfrak{o}(3,1)$ as the $D=3$ de Sitter algebra and calculate the contraction limit $R\to\infty$ (R is the de Sitter radius) providing an explicit Hopf algebra structure for the quantum deformation of the $D=3$ Poincaré algebra (with mass-like deformation parameters), which is the two-parameter light-cone κ-deformation of the $D=3$ Poincaré symmetry.

1 Introduction

In the last decade there have been considered quantum field theories on noncommutative space-time, after it had been shown that the noncommutativity might follow from quantum gravity corrections (see e.g. [1, 2]) or an open string theory with anti-symmetric tensor field background (see e.g. [3]). For simple examples of noncommutativity, in particular Heisenberg-like space-time commutators ($\theta_{\mu\nu}=\text{const}$), it has been shown [4–9] that the theory with noncommutative space-time is covariant under twisted Poincaré symmetry. In such an approach the violation of classical Poincaré invariance, e.g., by a constant tensor $\theta_{\mu\nu}$, can be equivalently described by twisting of the classical symmetry [8]. Because twist quantizations of the Lorentz and Poincaré algebras are classified by classical r-matrices satisfying homogeneous Yang–Baxter (YB) equations, it is interesting to approach the problem of possible violations of Lorentz and Poincaré symmetries by considering new, in particular non-Abelian, twist quantizations[1].

All quantum deformations of relativistic symmetries are described by Hopf-algebraic deformations of Lorentz and Poincaré algebras. Such quantum deformations determine infinitesimally Lorentz and Poincaré Poisson structures. These Poisson structures are described by classical r-matrices satisfying homogeneous as well as inhomogeneous (modified) YB equations, and they have been classified already some time ago by Zakrzewski (see [10] for the Lorentz classical r-matrices and [11] for the Poincaré classical matrices). In [10] there are provided four classical $\mathfrak{o}(3,1)$ r-matrices and in [11] one finds 21 cases describing different deformations of the Poincaré symmetry, with various numbers of free parameters.

In this paper we would like to describe the explicit Hopf algebra for an important nonstandard deformation of the $D=4$ Lorentz algebra generated by a two-parameter Jordanian classical r-matrix.

Part of the results presented in this paper have been given in our short report [12], where the complex algebra basis was used. In contrast to [12], here all obtained results are represented in a real (more physical) basis of the Lorentz algebra. In this case a twist two-tensor depends on new generators σ and φ which are operator analogs of polar coordinates for a complex plane. Moreover, we interpret the $D=4$ Lorentz algebra as a $D=3$ de Sitter algebra and, following the quantum contraction method applied firstly to the q-deformed $D=4$ anti-de Sitter (AdS) algebra [13], we calculate the contraction limit $R\to\infty$ (R is the de Sitter radius) providing a deformation of the $D=3$ Poincaré algebra. Subsequently, we present in explicit form the Hopf structure describing the two-parameter extension of the light-cone κ-deformation for the $D=3$ Poincaré symmetry.

The plan of this paper is as follows. In Sect. 2 we describe different bases of the real Lorentz algebra, $\mathfrak{o}(3,1)$, and we recall the classification of all possible quantum deformations of the $D=4$ Lorentz algebra [11]. In Sect. 3 we calculate the twist function corresponding to the non-Abelian Jordanian classical r-matrix (17) and describe ex-

[a] e-mail: lukier@ift.uni.wroc.pl

[1] By non-Abelian twist we mean a twist two-tensor with a support in a non-Abelian algebra.

plicitly the deformed co-products and antipodes in a classical $\mathfrak{o}(3,1)$ basis. In Sect. 4 we obtain a deformed $D=4$ Poincaré–Hopf algebra by adding four-momentum generators and calculating their twisted co-products. In Sect. 5 we rewrite our quantum deformation in terms of the $D=3$ de Sitter algebra and we perform the contraction, providing a two-parameter light-cone κ-deformation of the $D=3$ Poincaré symmetry. In Sect. 6 we provide possible extensions of the present work.

2 $D=4$ Lorentz algebra and its classical r-matrices

The classical canonical basis of the $D=4$ Lorentz algebra, $\mathfrak{o}(3,1)$, can be described by six anti-hermitian generators $(h,\ e_\pm,\ h',\ e'_\pm)$ satisfying the following nonvanishing commutation relations:

$$[h,e_\pm]=\pm e_\pm\,,\quad [e_+,e_-]=2h\,, \tag{1}$$

$$[h,e'_\pm]=\pm e'_\pm\,,\quad [h',e_\pm]=\pm e'_\pm\,,\quad [e_\pm,e'_\mp]=\pm 2h'\,, \tag{2}$$

$$[h',e'_\pm]=\mp e_\pm\,,\quad [e'_+,e'_-]=-2h\,, \tag{3}$$

and moreover

$$x^*=-x\quad (\forall x\in\mathfrak{o}(3,1))\,. \tag{4}$$

The formulas (1)–(3) can be rewritten as the following three $\mathfrak{o}(3)$ covariant relations describing a Lorentz algebra in a "physical" basis $(i,j,k=1,2,3)$[2]:

$$[M_i,M_j]=\mathrm{i}\varepsilon_{ijk}M_k\,,\quad [M_i,N_j]=\mathrm{i}\varepsilon_{ijk}N_k\,,$$
$$[N_i,N_j]=-\mathrm{i}\varepsilon_{ijk}M_k\,, \tag{5}$$

where the three-dimensional rotation generators M_i and boosts N_i are related with the canonical basis (1)–(3) as follows:

$$h=\mathrm{i}N_3\,,\qquad e_\pm=\mathrm{i}(N_1\pm M_2)\,,$$
$$h'=-\mathrm{i}M_3\,,\quad e'_\pm=\mathrm{i}(\pm N_2-M_1)\,, \tag{6}$$

and they are hermitian:

$$M_i^*=M_i\,,\quad N_i^*=N_i\quad (i=1,2,3)\,. \tag{7}$$

It should be stressed that the realization (6) of the canonical basis in terms of the physical generators is not unique. Indeed, it is easy to see that the formulas (5) are invariant with respect to the cyclic permutation of the indexes $1\to2\to3\to1$ for the generators M_i and N_i. Therefore, if we apply such cyclic replacements to the generators of the right side of (6), we obtain another physical assignments of the canonical basis. In order to obtain a suitable contraction limit (see Sect. 4) we perform the cyclic replacements two times and we get another physical assignment of the basis (1)–(3):

$$h=\mathrm{i}N_2\,,\qquad e_\pm=\mathrm{i}(N_3\pm M_1)\,,$$
$$h'=-\mathrm{i}M_2\,,\quad e'_\pm=\mathrm{i}(\pm N_1-M_3)\,. \tag{8}$$

The complete classification of the $D=4$ Lorentz quantum algebras in [10] is provided by the following list of the corresponding classical r-matrices (see also [14]). 1. Standard r-matrices related with realification of Drinfeld–Jimbo deformation of $\mathfrak{sl}(2,\mathbb{C})$ [15–17]

$$r_1(\alpha,\beta,\gamma)=\alpha\left(e'_+\wedge e_-+e_+\wedge e'_-\right)+\beta\left(e_+\wedge e_--e'_+\wedge e'_-\right)$$
$$+\gamma h\wedge h'\,, \tag{9}$$

$$r_2(\alpha)=\alpha\left(e'_+\wedge e_-+e_+\wedge e'_-+\frac{1}{2}h\wedge h'\right)\pm e_+\wedge e'_+\,. \tag{10}$$

2. Nonstandard r-matrices, related with realification of Jordanian deformation of $\mathfrak{sl}(2,\mathbb{C})$ [18–20]

$$r_3(\alpha,\beta)=\alpha(h\wedge e_+-h'\wedge e'_+)+\beta e_+\wedge e'_+\,, \tag{11}$$

$$r_4(\alpha)=\alpha h\wedge e_+\,. \tag{12}$$

Further we shall assume that the r-matrices are anti-hermitian, i.e. $r_i^*=-r_i$; therefore, all the parameters in (9)–(12) are purely imaginary.

The standard quantum deformations (9) and (10) do satisfy the modified YB equation and cannot be extended to the Poincaré algebra [21,22]. The nonstandard quantum deformations (11) and (12) can be extended to the whole Poincaré algebra and provide a new deformation of the relativistic symmetries. Because the quantization of (12) by means of the Ogievetsky twist [19] is straightforward and describes the well-known Jordanian deformation, we shall consider here more in detail only the two-parameter deformation generated by the r-matrix (11).

Further we shall employ as well the complex basis of the Lorentz algebra $(\mathfrak{o}(3,1)\simeq\mathfrak{o}(3;\mathbb{C})\oplus\bar{\mathfrak{o}}(3,\mathbb{C}))$ described by two commuting sets of complex generators:

$$H_1=\frac{1}{2}(h+\mathrm{i}h')\,,\quad E_{1\pm}=\frac{1}{2}\left(e_\pm+\mathrm{i}e'_\pm\right)\,, \tag{13}$$

$$H_2=\frac{1}{2}(h-\mathrm{i}h')\,,\quad E_{2\pm}=\frac{1}{2}\left(e_\pm-\mathrm{i}e'_\pm\right)\,, \tag{14}$$

which satisfy the relations (compare with (1))

$$[H_i,E_{i\pm}]=\pm E_{i\pm}\,,\quad [E_{i+},E_{i-}]=2H_i\quad (i=1,2)\,, \tag{15}$$

where the sets $(H_1,E_{1\pm})$ and $(H_2,E_{2\pm})$ do commute mutually. The $*$-operation describing the real structure acts on the generators H_i, and $E_{i\pm}$ $(i=1,2)$ as follows:

$$H_1^*=-H_2\,,\quad E_{1\pm}^*=-E_{2\pm}\,,$$
$$H_2^*=-H_1\,,\quad E_{2\pm}^*=-E_{1\pm}\,. \tag{16}$$

[2] In what follows the symbol "i" means the imaginary unit, $\mathrm{i}^2=-1$.

The classical r-matrix $r_3(\alpha,\beta)$ (see (11)) in the complex basis of (13) and (14) takes the form

$$r_3(\alpha,\beta) = r'_3(\alpha) + r''_3(\beta)\,, \tag{17}$$
$$r'_3(\alpha) := 2\alpha(H_1\wedge E_{1+} + H_2\wedge E_{2+})\,,$$
$$r''_3(\beta) := 2\mathrm{i}\beta E_{1+}\wedge E_{2+}\,. \tag{18}$$

3 Two-parameter nonstandard deformation of $\mathfrak{o}(3,1)$

The first classical r-matrix, $r'_3(\alpha)$ in (17), is a sum of two Jordanian classical r-matrices, $\alpha H_1\wedge E_1$ and $\alpha H_2\wedge E_2$, which mutually commute. From this property follows that the first twisting two-tensor F' corresponding to the Jordanian type r-matrix $r'_3(\alpha)$ is a product of two Jordanian twists with the same deformation parameter α:

$$F' := F_{1J}F_{2J} = F_{2J}F_{1J} = \exp\left(H_1\otimes\sigma_1 + H_2\otimes\sigma_2\right)\,, \tag{19}$$

where

$$\sigma_i = \ln\left(1+2\alpha E_{i+}\right)\,,\quad i=1,2\,. \tag{20}$$

It should be noted that the Jordanian two-tensor (19) is $*$-unitary, i.e.

$$F'^{*} = F'^{-1}\,. \tag{21}$$

Since after twisting by the two-tensor (19) the generators (20) have the primitive co-products

$$\Delta^{(F')}(\sigma_i) = \sigma_i\otimes 1 + 1\otimes\sigma_i\,,\quad i=1,2\,; \tag{22}$$

therefore, the two-tensor[3,4]

$$F'' = \exp\left(\mathrm{i}\frac{\beta}{2\alpha^2}\sigma_1\wedge\sigma_2\right) \tag{23}$$

satisfies the cocycle condition [23]

$$F''^{12}(\Delta^{(F')}\otimes\mathrm{id})(F'') = F''^{23}(\mathrm{id}\otimes\Delta^{(F')})(F'')\,, \tag{24}$$

and the "unital" normalization condition

$$(\varepsilon\otimes\mathrm{id})(F'') = (\mathrm{id}\otimes\varepsilon)(F'') = 1\,. \tag{25}$$

Thus the complete twisting two-tensor $F(\alpha,\beta)$ corresponding to the Jordanian type r-matrix (17) is given as follows:

$$\begin{aligned} F &:= F(\alpha,\beta) = F''F' \\ &= \exp\left(\mathrm{i}\frac{\beta}{2\alpha^2}\sigma_1\wedge\sigma_2\right)\exp\left(H_1\otimes\sigma_1 + H_2\otimes\sigma_2\right)\,. \end{aligned} \tag{26}$$

[3] In the limit $\alpha\to 0$ the two-tensor F'' goes to $\exp\left(2\mathrm{i}\beta E_{1+}\wedge E_{2+}\right)$ which is the twist in the direction of $r''_3(\alpha=0,\beta)$ (see (11)).

[4] It should be mentioned that modulo the explicit parameters dependence the algebraic form of the twist (23) was early pointed out by Kulish and Mudrov (see [24], item 3 on the list on p. 6).

Let us express this function in terms of the generators $(h,\ e_\pm,\ h',\ e'_\pm)$. Firstly we notice that the generators σ_i given by (20) do not change after the second twist F'', i.e. they have the primitive co-products

$$\Delta^{(F)}(\sigma_i) = F''\Delta^{(F')}(\sigma_i)F''^{-1} = \sigma_i\otimes 1 + 1\otimes\sigma_i\,,\quad i=1,2\,. \tag{27}$$

Therefore, if instead of σ_1 and σ_2 we introduce the new generators σ and φ by the formulas

$$\sigma_1 = \sigma + \mathrm{i}\varphi\,,\quad \sigma_2 = \sigma - \mathrm{i}\varphi\,, \tag{28}$$

where

$$\sigma = \frac{1}{2}\ln\left[(1+\alpha e_+)^2 + (\alpha e'_+)^2\right]\,,\quad \varphi = \arctan\frac{\alpha e'_+}{1+\alpha e_+}\,, \tag{29}$$

then the new generators also have the primitive co-products

$$\Delta^{(F)}(\sigma) = \sigma\otimes 1 + 1\otimes\sigma\,,\quad \Delta^{(F)}(\varphi) = \varphi\otimes 1 + 1\otimes\varphi\,, \tag{30}$$

and moreover they are $*$-hermitian:

$$\sigma^* = \sigma\,,\quad \varphi^* = \varphi\,. \tag{31}$$

The formulas inverse to (29) have the form

$$\alpha e_+ = \mathrm{e}^{\sigma}\cos\varphi - 1\,,\quad \alpha e'_+ = \mathrm{e}^{\sigma}\sin\varphi\,. \tag{32}$$

Substituting (28), (13) and (14) in (26) we obtain the following formula for the twist two-tensor in terms of the canonical $\mathfrak{o}(3,1)$ basis (1)–(3):

$$F(\alpha,\beta) = \exp\left(\frac{\beta}{\alpha^2}\sigma\wedge\varphi\right)\exp\left(h\otimes\sigma - h'\otimes\varphi\right)\,. \tag{33}$$

This two-tensor is $*$-unitary, i.e.

$$F^*(\alpha,\beta) = F^{-1}(\alpha,\beta)\,. \tag{34}$$

Deformed co-products for the canonical $\mathfrak{o}(3,1)$ basis can be obtained in two ways. The first way is to apply the twisting two-tensor in the form (26) to the trivial co-products of the complex generators H_i, E_i, F_i $(i=1,2)$ and then using the formulas (13) and (14) to derive the co-products and antipodes for the canonical basis. The other way is to apply the twisting two-tensor in the form (33) directly to the trivial co-products of the canonical generators (1) and (2), $(\Delta_{\alpha,\beta}(\cdot) := F\Delta(\cdot)F^{-1})$. For convenience we shall use the notation $c_\varphi := \cos\varphi$ and $s_\varphi := \sin\varphi$. We obtain the following formulae:

$$\begin{aligned}\Delta_{\alpha,\beta}(h) &= h\otimes \mathrm{e}^{-\sigma}c_\varphi + 1\otimes h + h'\otimes \mathrm{e}^{-\sigma}s_\varphi \\ &\quad + \frac{\beta}{\alpha^2}\big(\mathrm{e}^{-\sigma}s_\varphi\otimes \mathrm{e}^{-\sigma}(\sigma c_\varphi-\varphi s_\varphi) \\ &\quad + (\mathrm{e}^{-\sigma}c_\varphi-1)\otimes \mathrm{e}^{-\sigma}(\varphi c_\varphi+\sigma s_\varphi) - \sigma\otimes \mathrm{e}^{-\sigma}s_\varphi \\ &\quad -\varphi\otimes(\mathrm{e}^{-\sigma}c_\varphi-1)\big)\,, \end{aligned} \tag{35}$$

$$\begin{aligned}\Delta_{\alpha,\beta}(h') &= h'\otimes \mathrm{e}^{-\sigma}c_\varphi + 1\otimes h' - h\otimes \mathrm{e}^{-\sigma}s_\varphi \\ &\quad + \frac{\beta}{\alpha^2}\big(-\mathrm{e}^{-\sigma}s_\varphi\otimes \mathrm{e}^{-\sigma}(\sigma s_\varphi+\varphi c_\varphi) \\ &\quad + (\mathrm{e}^{-\sigma}c_\varphi-1)\otimes \mathrm{e}^{-\sigma}(\sigma c_\varphi-\varphi s_\varphi) + \varphi\otimes \mathrm{e}^{-\sigma}s_\varphi \\ &\quad -\sigma\otimes(\mathrm{e}^{-\sigma}c_\varphi-1)\big)\,. \end{aligned} \tag{36}$$

The remaining co-products $\Delta_{\alpha,\beta}(e_-)$ and $\Delta_{\alpha,\beta}(e'_-)$ are given by lengthy formulae, and therefore we present the β-dependent terms using the complex basis:

$$\begin{aligned}\Delta_{\alpha,\beta}(e_-) &= e_-\otimes \mathrm{e}^{-\sigma}c_\varphi + 1\otimes e_- + e'_-\otimes \mathrm{e}^{-\sigma}s_\varphi \\ &\quad + \alpha\big(h\otimes(\{h',\mathrm{e}^{-\sigma}c_\varphi\}+\{h',\mathrm{e}^{-\sigma}s_\varphi\}) \\ &\quad - h'\otimes(\{h',\mathrm{e}^{-\sigma}c_\varphi\}-\{h,\mathrm{e}^{-\sigma}s_\varphi\}) \\ &\quad + (h^2-h'^2)\otimes(\mathrm{e}^{-2\sigma}c_{2\varphi}-\mathrm{e}^{-\sigma}c_\varphi) \\ &\quad +2hh'\otimes(\mathrm{e}^{-2\sigma}s_{2\varphi}-\mathrm{e}^{-\sigma}s_\varphi)\big) \\ &\quad + \frac{\mathrm{i}\beta}{4\alpha}(\mathcal{E}_1+\mathcal{E}_1^*) - \frac{\beta^2}{4\alpha^3}(\mathcal{E}_2+\mathcal{E}_2^*)\,, \end{aligned} \tag{37}$$

$$\begin{aligned}\Delta_{\alpha,\beta}(e'_-) &= e'_-\otimes \mathrm{e}^{-\sigma}c_\varphi + 1\otimes e'_- + e_-\otimes \mathrm{e}^{-\sigma}s_\varphi \\ &\quad + \alpha\big(h'\otimes(\{h,\mathrm{e}^{-\sigma}c_\varphi\}+\{h',\mathrm{e}^{-\sigma}s_\varphi\}) \\ &\quad + h\otimes(\{h',\mathrm{e}^{-\sigma}c_\varphi\}-\{h,\mathrm{e}^{-\sigma}s_\varphi\}) \\ &\quad - (h^2-h'^2)\otimes(\mathrm{e}^{-2\sigma}s_{2\varphi}-\mathrm{e}^{-\sigma}s_\varphi) \\ &\quad +2hh'\otimes(\mathrm{e}^{-2\sigma}c_{2\varphi}-\mathrm{e}^{-\sigma}c_\varphi)\big) \\ &\quad + \frac{\beta}{4\alpha}(\mathcal{E}_1-\mathcal{E}_1^*) + \frac{\mathrm{i}\beta^2}{4\alpha^3}(\mathcal{E}_2-\mathcal{E}_2^*)\,, \end{aligned} \tag{38}$$

where

$$\begin{aligned}\mathcal{E}_1 &= \{\tilde H_1,\mathrm{e}^{-\sigma_1}\}\otimes\sigma_2\mathrm{e}^{-\sigma_1} - \sigma_2\otimes\{\tilde H_1,\mathrm{e}^{-\sigma_1}\} \\ &\quad + \Lambda_1\otimes\{\tilde H_1,\mathrm{e}^{-\sigma_1}\}\sigma_2 - \{\tilde H_1,\Lambda_1\}\otimes\sigma_2\Lambda_1\mathrm{e}^{-\sigma_1} \\ &\quad - \{\tilde H_1,\sigma_2\}\otimes\Lambda_1\mathrm{e}^{-\sigma_1}\,, \end{aligned} \tag{39}$$

$$\begin{aligned}\mathcal{E}_2 &= \sigma_2^2\otimes\Lambda_1\mathrm{e}^{-\sigma_1} + \Lambda_1\mathrm{e}^{-\sigma_1}\otimes\sigma_2^2\mathrm{e}^{-\sigma_1} + \Lambda_1^2\otimes\sigma_2^2\Lambda_1\mathrm{e}^{-\sigma_1} \\ &\quad - 2\Lambda_1\sigma_2\otimes\sigma_2\Lambda_1\mathrm{e}^{-\sigma_1}\,, \end{aligned} \tag{40}$$

and

$$\begin{aligned}\tilde H_1 &= h+\mathrm{i}h'\,, \quad \Lambda_1 = \mathrm{e}^{-\sigma_1}-1\,, \\ \sigma_1 &= \sigma+\mathrm{i}\varphi\,, \quad \sigma_2 = \sigma-\mathrm{i}\varphi\,. \end{aligned} \tag{41}$$

Here by the brackets $\{\cdot,\cdot\}$ we mean the anti-commutator $\{a,b\}=ab+ba$.

Explicit formulae for the antipodes are given by

$$S_{\alpha,\beta}(\sigma) = -\sigma\,, \quad S_{\alpha,\beta}(\varphi) = -\varphi\,, \tag{42}$$

$$\begin{aligned} S_{\alpha,\beta}(h) &= -h\mathrm{e}^{\sigma}c_\varphi + h'\mathrm{e}^{\sigma}s_\varphi\,, \\ S_{\alpha,\beta}(h') &= -h'\mathrm{e}^{\sigma}c_\varphi - h\mathrm{e}^{\sigma}s_\varphi\,, \end{aligned} \tag{43}$$

$$\begin{aligned} S_{\alpha,\beta}(e_-) &= -e_-\mathrm{e}^{\sigma}c_\varphi + e'_-\mathrm{e}^{\sigma}s_\varphi \\ &\quad + \alpha(h^2-h'^2)(\mathrm{e}^{2\sigma}c_{2\varphi}+\mathrm{e}^{\sigma}c_\varphi) \\ &\quad - 2\alpha hh'(\mathrm{e}^{2\sigma}s_{2\varphi}+\mathrm{e}^{\sigma}s_\varphi)\,, \end{aligned} \tag{44}$$

$$\begin{aligned} S_{\alpha,\beta}(e'_-) &= -e'_-\mathrm{e}^{\sigma}c_\varphi - e_-\mathrm{e}^{\sigma}s_\varphi \\ &\quad + \alpha(h^2-h'^2)(\mathrm{e}^{2\sigma}s_{2\varphi}+\mathrm{e}^{\sigma}s_\varphi) \\ &\quad + 2\alpha hh'(\mathrm{e}^{2\sigma}c_{2\varphi}+\mathrm{e}^{\sigma}c_\varphi)\,. \end{aligned} \tag{45}$$

Using the relations (6) the formulae (35)–(45) can be rewritten in terms of the physical generators M_j and N_j ($j=1,2,3$).

4 Extension of the deformation to Poincaré algebra

The $D=4$ Lorentz algebra (6) can be extended to the $D=4$ Poincaré algebra by adding the mutually commuting four-momentum operators (P_0,P_1,P_2,P_3) satisfying the relations $(j,k,l=1,2,3)$

$$\begin{aligned} [M_i,P_j] &= \mathrm{i}\varepsilon_{ijk}P_k\,, \qquad [M_i,P_0]=0\,, \\ [N_i,P_j] &= -\mathrm{i}\delta_{ij}P_0\,, \qquad [N_i,P_0] = -\mathrm{i}P_i\,. \end{aligned} \tag{46}$$

The formulas (46) say that a linear span of the four-momentum generators P_μ, $(\mu=0,1,2,3)$ is a four-dimensional $\mathfrak{o}(3,1)$-module with respect to the adjoint action of the Lorentz algebra on the four-momentum space. In order to describe in this space the actions of the canonical basis $(h,e_\pm,h',e'_\pm)$ it is useful to use as a technical tool (see also [11]) the matrix realization of the four-momenta. The matrix realization is constructed as follows.

First we introduce the following basis in the four-momentum space:

$$\begin{aligned} P_{11} &= \frac{1}{2}(P_0+P_3)\,, \qquad P_{12} = \frac{1}{2}(P_1-\mathrm{i}P_2)\,, \\ P_{21} &= \frac{1}{2}(P_1+\mathrm{i}P_2)\,, \qquad P_{22} = \frac{1}{2}(P_0-P_3)\,. \end{aligned} \tag{47}$$

Let $\hat P_{ij}$, $(i,j=1,2)$ be the 2×2 matrices $(\hat P_{ij})_{kl}=\delta_{ik}\delta_{jl}$. The matrix realization of the four-momenta is given by $P_{ij}\to\hat P_{ij}$. The arbitrary four-vector $\mathbf{P} := \sum_{i,j=1,2}\xi_{ij}P_{ij}$ ($\xi_{ij}\in\mathbb{R}$) is represented by the general 2×2 matrix

$$\mathbf{P}\to\hat{\mathbf{P}} := \sum_{i,j=1,2}\xi_{ij}\hat P_{ij} = \begin{pmatrix}\xi_{11} & \xi_{12}\\ \xi_{21} & \xi_{22}\end{pmatrix}, \tag{48}$$

which will be used below to describe the transformation properties of the generators (47) in a compact way. The corresponding 2×2 matrix realization of the Lorentz algebra (1)–(3) will be used:

$$h \to H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$e_+ \to E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_- \to E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{49}$$
$$h' \to H' = \mathrm{i}H\,, \quad e'_+ \to E'_+ = \mathrm{i}E_+\,, \quad e'_- \to E'_- = \mathrm{i}E_-\,. \tag{50}$$

The action of any element $x \in (h, e_\pm, h', e'_\pm)$ on the matrix-vector $\hat{\mathbf{P}}$ is given by [11]

$$[x, \mathbf{P}] \to x \triangleright \hat{\mathbf{P}} := X\hat{\mathbf{P}} + \hat{\mathbf{P}}X^+\,, \tag{51}$$

where X^+ is the hermitian conjugate of X. Using the formula (51) we find, for example,

$$h \triangleright \hat{\mathbf{P}} = \begin{pmatrix} \xi_{11} & 0 \\ 0 & -\xi_{22} \end{pmatrix}, \qquad h' \triangleright \hat{\mathbf{P}} = \mathrm{i}\begin{pmatrix} 0 & \xi_{21} \\ -\xi_{12} & 0 \end{pmatrix}, \tag{52}$$
$$e_+ \triangleright \hat{\mathbf{P}} = \begin{pmatrix} \xi_{21}+\xi_{12} & \xi_{22} \\ \xi_{22} & 0 \end{pmatrix}, \quad e'_+ \triangleright \hat{\mathbf{P}} = \mathrm{i}\begin{pmatrix} \xi_{21}-\xi_{12} & \xi_{22} \\ -\xi_{22} & 0 \end{pmatrix}. \tag{53}$$

The classical r-matrix (11) for the $D=4$ Lorentz algebra satisfies the classical YB equation and provides also the deformation of the $D=4$ Poincaré algebra. The modification of the classical Hopf algebra structure is described by the twist function (33). Let us calculate the twisted co-products of the four-momenta in our 2×2 matrix realization (48) using the restricted twist two-tensor $F'(\alpha) := F(\alpha, \beta=0)$. We have

$$\Delta_\alpha(\hat{\mathbf{P}}) = F'(\alpha)\hat{\mathbf{P}} \otimes 1 F'^{-1}(\alpha) + F'(\alpha) 1 \otimes \hat{\mathbf{P}} F'^{-1}(\alpha)\,. \tag{54}$$

By employing the formulas (51)–(53) we find

$$\begin{aligned}\Delta_\alpha(\hat{\mathbf{P}}) &= \begin{pmatrix} \xi_{11}\otimes \mathrm{e}^{\sigma} & \xi_{12}\otimes \mathrm{e}^{\mathrm{i}\varphi} \\ \xi_{21}\otimes \mathrm{e}^{-\mathrm{i}\varphi} & \xi_{22}\otimes \mathrm{e}^{-\sigma} \end{pmatrix} + 1\otimes \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix} \\ &\quad + \alpha h \otimes \left\{ \mathrm{e}^{-\sigma+\mathrm{i}\varphi} \begin{pmatrix} \xi_{21} & \xi_{22} \\ 0 & 0 \end{pmatrix} + \mathrm{e}^{-\sigma-\mathrm{i}\varphi} \begin{pmatrix} \xi_{12} & 0 \\ \xi_{22} & 0 \end{pmatrix} \right\} \\ &\quad - \mathrm{i}\alpha h' \otimes \left\{ \mathrm{e}^{-\sigma+\mathrm{i}\varphi} \begin{pmatrix} \xi_{21} & \xi_{22} \\ 0 & 0 \end{pmatrix} - \mathrm{e}^{-\sigma-\mathrm{i}\varphi} \begin{pmatrix} \xi_{12} & 0 \\ \xi_{22} & 0 \end{pmatrix} \right\} \\ &\quad + \alpha^2 (h^2 + h'^2) \otimes \mathrm{e}^{-2\sigma} \begin{pmatrix} \xi_{22} & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{55}$$

Because $\partial\hat{\mathbf{P}}/\partial\xi_{ij} = \hat{P}_{ij}$, by differentiating (55) we obtain the formulas for the twisted co-products of the four-momentum components $\hat{P}_{ij}$. These formulas after replacing the realizations $\hat{P}_{ij}$ by the initial generators P_{ij} take the following compact form:

$$\begin{aligned}&\begin{pmatrix} \Delta_\alpha(P_{11}) & \Delta_\alpha(P_{12}) \\ \Delta_\alpha(P_{21}) & \Delta_\alpha(P_{22}) \end{pmatrix} \\ &= \begin{pmatrix} P_{11}\otimes \mathrm{e}^{\sigma} & P_{12}\otimes \mathrm{e}^{-\mathrm{i}\varphi} \\ P_{21}\otimes \mathrm{e}^{\mathrm{i}\varphi} & P_{22}\otimes \mathrm{e}^{-\sigma} \end{pmatrix} + 1\otimes \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \\ &\quad + \alpha h \otimes \begin{pmatrix} 0 & \mathrm{e}^{-\sigma-\mathrm{i}\varphi}P_{11} \\ \mathrm{e}^{-\sigma+\mathrm{i}\varphi}P_{11} & \mathrm{e}^{-\sigma+\mathrm{i}\varphi}P_{12} + \mathrm{e}^{-\sigma-\mathrm{i}\varphi}P_{21} \end{pmatrix} \\ &\quad - \mathrm{i}\alpha h' \otimes \begin{pmatrix} 0 & -\mathrm{e}^{-\sigma-\mathrm{i}\varphi}P_{11} \\ \mathrm{e}^{-\sigma+\mathrm{i}\varphi}P_{11} & \mathrm{e}^{-\sigma+\mathrm{i}\varphi}P_{12} - \mathrm{e}^{-\sigma-\mathrm{i}\varphi}P_{21} \end{pmatrix} \\ &\quad + \alpha^2(h^2 + h'^2) \otimes \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{e}^{-2\sigma}P_{11} \end{pmatrix}. \end{aligned} \tag{56}$$

In an analogous way we can calculate the complete twisted co-products of the four-momenta using the general twist two-tensor $F(\alpha, \beta)$, but these formulae are lengthy.

5 Two-parameter light-cone deformation of the $D=3$ Poincaré algebra

The $D=4$ Lorentz algebra $\mathfrak{o}(3,1)$ (see (5)) can be reinterpreted as describing the $D=3$ de Sitter algebra where the generators N_1, N_2 describe two $D=3$ boosts, M_3 generates $\mathfrak{o}(2)$ rotations, and three curved translations are generated by M_1, M_2 and N_3. Therefore the formulae (35)–(45) can be used for the description of the quantum Hopf-algebraic deformations of the $D=3$ de Sitter algebra. Below we shall use the formulae (8) relating the physical Lorentz generators with the mathematical basis (1)–(3). In this realization the classical r-matrix (11) has the form

$$\begin{aligned} r_3(\alpha,\beta) &= -\alpha\left(N_2 \wedge (N_3 + M_1) + M_2 \wedge (N_1 - M_3)\right) \\ &\quad - \beta(N_3 + M_1)\wedge(N_1 - M_3)\,. \end{aligned} \tag{57}$$

Introducing the standard dS rescaling $M_1 = R\mathcal{P}_2$, $M_2 = -R\mathcal{P}_1$, $N_3 = R\mathcal{P}_0$, we obtain for the r-matrix (57) the following form:

$$\begin{aligned} r_3(\alpha,\beta) &= -\alpha R\left(N_2 \wedge (\mathcal{P}_0 + \mathcal{P}_2) + \mathcal{P}_1 \wedge (M_3 - N_1)\right) \\ &\quad - \beta R(\mathcal{P}_0 + \mathcal{P}_2)\wedge(M_3 - N_1)\,. \end{aligned} \tag{58}$$

Now we put $\alpha = -\mathrm{i}/\kappa R$, $\beta = -\mathrm{i}/\kappa' R$ (where κ and κ' are real mass-like parameters) and perform the limit $R\to\infty$. In such a way we obtain the classical r-matrix for the $D=3$ Poincaré algebra ($\lim_{R\to\infty} \mathcal{P}_\mu = P_\mu$, $\mu = 0,1,2$):

$$\begin{aligned} r_{\kappa,\kappa'} &:= \lim_{R\to\infty} r_3\left(\frac{-\mathrm{i}}{\kappa R}, \frac{-\mathrm{i}}{\kappa' R}\right) \\ &= \frac{\mathrm{i}}{\kappa}\left(N_2 \wedge (P_0 + P_2) + P_1 \wedge (M_3 - N_1)\right) \\ &\quad + \frac{\mathrm{i}}{\kappa'}(P_0 + P_2)\wedge(M_3 - N_1)\,, \end{aligned} \tag{59}$$

where the parameters κ and κ' are new dimensionful deformation parameters. The $D=3$ Poincaré algebra is described by the generators M_3, N_1, N_2 of the $D=3$ Lorentz algebra $\mathfrak{o}(2,1)$, and the generators P_0, P_1, P_2 span the three-momentum sector.

The classical r-matrix (59) describes a two-parameter light-cone κ-deformation of the $D=3$ Poincaré algebra[5]. Using the commutation relations for the generators M_3, N_1, N_2, $P_\pm := P_0 \pm P_2$, it is easy to check that the r-matrix $r_{\kappa,\kappa'}$, when κ' goes to ∞, is of Jordanian type[6]. Therefore using known general formulae [26, 27] we can immediately write down the twisting two-tensor corresponding to the r-matrix $r_{\kappa,\kappa'}$. However we can obtain also a twisting two-tensor corresponding to the classical r-matrix (59) by applying the dS contraction limit to the full twisting two-tensor (33). First of all it is easy to obtain in the limit $R\to\infty$ the following formulas:

$$\begin{aligned}\sigma_+ &:= \lim_{R\to\infty}\sigma \\ &= \lim_{R\to\infty}\frac{1}{2}\ln\left[\left(1+\frac{1}{\kappa}\mathcal{P}_+\right)^2+\frac{1}{\kappa^2R^2}\left(N_1+M_3\right)^2\right] \\ &= \ln\left(1+\frac{1}{\kappa}P_+\right)\,, \end{aligned} \tag{60}$$

$$\begin{aligned}\lim_{R\to\infty}(R\varphi) &= \lim_{R\to\infty} R\arctan\frac{(N_1-M_3)/\kappa R}{(1+\mathcal{P}_+/\kappa)} \\ &= \frac{1}{\kappa}(N_1-M_3)\mathrm{e}^{-\sigma_+}\,. \end{aligned} \tag{61}$$

Using the formulas (60) and (61) we get the twist two-tensor $F_{\kappa,\kappa'}$ corresponding to the classical r-matrix (59):

$$\begin{aligned}F_{\kappa,\kappa'} &= \lim_{R\to\infty} F\left(\frac{-\mathrm{i}}{\kappa R},\frac{-\mathrm{i}}{\kappa' R}\right) \\ &:= \exp\left(\frac{\mathrm{i}\kappa}{\kappa'}\sigma_+\otimes(N_1-M_3)\mathrm{e}^{-\sigma_+}\right) \\ &\quad\times\exp\left(\frac{\mathrm{i}}{\kappa}P_1\otimes(N_1-M_3)\mathrm{e}^{-\sigma_+}\right)\exp\left(\mathrm{i}N_2\otimes\sigma_+\right)\,. \end{aligned} \tag{62}$$

By applying the rescaling of the formulas (35)–(40) (introducing the dS radius R and performing the limit $R\to\infty$) we obtain the list of deformed co-products for all generators of the $D=3$ Poincaré algebra. For convenience in what follows we set $\Lambda_+ := \mathrm{e}^{-\sigma_+}-1$, $L_+ := N_1 - M_+$ and $L_- := N_1 + M_+$. The co-products are given by

$$\Delta_{\kappa,\kappa'}(\sigma_+) = \sigma_+\otimes 1 + 1\otimes\sigma_+\,, \tag{63}$$

$$\begin{aligned}\Delta_{\kappa,\kappa'}(P_1) &= P_1\otimes\mathrm{e}^{-\sigma_+}+1\otimes P_1 \\ &\quad+\frac{\kappa^2}{\kappa'}\left(\Lambda_+\otimes\sigma_+\mathrm{e}^{-\sigma_+}-\sigma_+\otimes\Lambda_+\right)\,, \end{aligned} \tag{64}$$

$$\begin{aligned}\Delta_{\kappa,\kappa'}(P_-) &= P_-\otimes\mathrm{e}^{-\sigma_+}+1\otimes P_- \\ &\quad-\frac{1}{\kappa}\left(2P_1\otimes P_1\mathrm{e}^{-\sigma_+}+P_1^2\otimes\Lambda_+\mathrm{e}^{-\sigma_+}\right) \\ &\quad-\frac{2\kappa}{\kappa'}\left(P_1\mathrm{e}^{-\sigma_+}\otimes\sigma_+\mathrm{e}^{-\sigma_+}-\sigma_+\otimes P_1\mathrm{e}^{-\sigma_+}\right. \\ &\quad-P_1\Lambda_+\otimes\sigma_+\Lambda_+\mathrm{e}^{-\sigma_+}-P_1\sigma_+\otimes\Lambda_+\mathrm{e}^{-\sigma_+} \\ &\quad\left.+\Lambda_+\otimes P_1\mathrm{e}^{-\sigma_+}\right) \\ &\quad-\frac{\kappa^3}{\kappa'^2}\left(\Lambda_+\mathrm{e}^{-\sigma_+}\otimes\sigma_+^2\mathrm{e}^{-\sigma_+}+\sigma_+^2\otimes\Lambda_+\mathrm{e}^{-\sigma_+}\right. \\ &\quad\left.+\Lambda_+^2\otimes\sigma_+^2\Lambda_+\mathrm{e}^{-\sigma_+}-2\Lambda_+\sigma_+\otimes\sigma_+\Lambda_+\mathrm{e}^{-\sigma_+}\right)\,, \end{aligned} \tag{65}$$

$$\Delta_{\kappa,\kappa'}(L_+) = L_+\otimes\mathrm{e}^{\sigma_+}+\mathrm{e}^{\sigma_+}\otimes L_+\,, \tag{66}$$

$$\begin{aligned}\Delta_{\kappa,\kappa'}(N_2) &= N_2\otimes\mathrm{e}^{-\sigma_+}+1\otimes N_2 \\ &\quad+\frac{1}{\kappa}P_1\otimes L_+\mathrm{e}^{-2\sigma_+} \\ &\quad-\frac{\kappa}{\kappa'}\left(L_+\mathrm{e}^{-\sigma_+}\otimes\Lambda_+-L_+\mathrm{e}^{-2\sigma_+}\otimes\sigma_+\mathrm{e}^{-\sigma_+}\right. \\ &\quad\left.-\Lambda_+\otimes L_+(\sigma_++1)\mathrm{e}^{-2\sigma_+}+\sigma_+\otimes L_+\mathrm{e}^{-2\sigma_+}\right)\,. \end{aligned} \tag{67}$$

The last co-product appears to be very lengthy and we present it only in the limit $\kappa'\to\infty$:

$$\begin{aligned}\Delta_\kappa(L_-) &= L_-\otimes\mathrm{e}^{-\sigma_+}+1\otimes L_- \\ &\quad+\frac{1}{\kappa}\left(P_-\otimes L_+\mathrm{e}^{-\sigma_+}-P_1\otimes\{N_2,\mathrm{e}^{-\sigma_+}\}\right. \\ &\quad\left.-2N_2\otimes P_1\mathrm{e}^{-\sigma_+}-2N_2P_1\otimes\Lambda_+\mathrm{e}^{-\sigma_+}\right) \\ &\quad-\frac{1}{\kappa^2}\left(P_1\otimes\{P_1,L_+\}\mathrm{e}^{-2\sigma_+}\right. \\ &\quad\left.+P_1^2\otimes L_+(2\Lambda_++1)\mathrm{e}^{-2\sigma_+}\right)\,. \end{aligned} \tag{68}$$

Using the formulas (41)–(45) one can calculate the antipodes

$$S_{\kappa,\kappa'}(\sigma_+) = -\sigma_+\,,\quad S_{\kappa,\kappa'}(P_1) = -P_1\mathrm{e}^{\sigma_+}\,, \tag{69}$$

$$S_{\kappa,\kappa'}(P_-) = -P_-\mathrm{e}^{\sigma_+}-\frac{1}{\kappa}P_1^2(\mathrm{e}^{2\sigma_+}+\mathrm{e}^{\sigma_+})\,, \tag{70}$$

$$S_{\kappa,\kappa'}(L_+) = -L_+\mathrm{e}^{-2\sigma_+}\,,\quad S_{\kappa,\kappa'}(N_2) = -N_2\mathrm{e}^{\sigma_+}+\frac{1}{\kappa}P_1L_+\,, \tag{71}$$

$$\begin{aligned}S_{\kappa,\kappa'}(L_-) &= -L_-\mathrm{e}^{\sigma_+}-\frac{1}{\kappa}\left(P_-L_++2N_2P_1(\mathrm{e}^{2\sigma_+}+\mathrm{e}^{\sigma_+})\right) \\ &\quad+\frac{1}{\kappa^2}P_1^2L_+(2\mathrm{e}^{\sigma_+}+1)\,. \end{aligned} \tag{72}$$

6 Outlook

In this paper we have described firstly in detail the Hopf algebra structure of the two-parameter twisted $D=4$ Lorentz algebras, which induce as well the twist quantization of the Poincaré algebra. It should be stressed that contrary to the cases of twisted Lorentz and Poincaré symmetries with Abelian twists which were considered recently in the literature [5–9] our deformation is generated by the non-Abelian twist. We also performed the dS contraction limit of the twisted $D=4$ Lorentz algebra and obtained the deformation of the $D=3$ Poincaré algebra with the explicit Hopf algebra structure.

A question which should be further addressed is the one to the Hopf structure of the dual quantum Poincaré group and the description of the corresponding noncommutative

5 For analogous considerations for the $D=4$ case see [25].

6 Compare with [26], formulae (2.12), after the assignment $h_\theta=\mathrm{i}N_2$, $e_\theta=P_+$, $e_{\gamma_1}=P_1$, $e_{\gamma_{-1}}=(M_3-N_1)$ (see also [25]).

Minkowski space. The quantum Poincaré groups generated by the classical Lorentz r-matrix (11) can be found on the list of deformed Poincaré groups given by Podleś and Woronowicz [22]. Another method providing a quantum algebra of noncommutative Poincaré group parameters is to calculate in the adjoint 4×4 matrix representation the quantum R-matrix and apply the RTT method [28]. In our case the universal R-matrix takes the form

$$\begin{aligned} R &= F^{21}(\alpha,\beta) F^{-1}(\alpha,\beta) \\ &= \exp\left(\frac{\beta}{\alpha^2}\varphi \wedge \sigma\right) \exp\left(\sigma \otimes h - \varphi \otimes h'\right) \\ &\quad \times \exp\left(h' \otimes \varphi - h \otimes \sigma\right) \exp\left(\frac{\beta}{\alpha^2}\varphi \wedge \sigma\right). \end{aligned} \tag{73}$$

Using the adjoint matrix representation of the Lorentz generators $(M_{ij})_{\mu\nu} = \delta_{\mu i}\delta_{\nu j} - \delta_{\mu j}\delta_{\nu i}$, $(M_{0j})_{\mu\nu} = \delta_{\mu 0}\delta_{\nu j} - \delta_{\mu j}\delta_{\nu 0}$ $(\mu,\nu = 0,1,2,3; i,j = 1,2,3)$ we can obtain from (73) a 16×16-dimensional quantum adjoint R-matrix for the Lorentz group. The R-matrix (73) can be treated also as the universal R-matrix for the $D = 4$ Poincaré algebra, and one can obtain easily the 25×25-dimensional quantum R-matrix for the $D = 4$ Poincaré algebra[7].

It is well known that one can obtain the $D = 4$ κ-deformed Poincaré algebra in its standard form (with time variable quantized) if we perform the quantum AdS contraction of the q-deformed AdS algebra $U_q(\mathfrak{so}(3,2))$ [13]. In Sect. 5 we have shown that if we treat the $D = 4$ Lorentz algebra as the de Sitter algebra one obtains by a suitable quantum dS contraction the generalized light-cone κ-deformation of the $D = 3$ Poincaré algebra, with two independent mass-like parameters κ and κ'. Therefore one can conclude that a quantum extension of the Inonu–Wigner contraction procedure to the twisted de Sitter (or anti-de Sitter) Hopf algebras can be used as the derivation method of twisted Poincaré symmetries.

Acknowledgements. The paper has been supported by KBN grant 1PO3B01828 (A.B., J.L.) and the grants RFBR-05-01-01086, INTAS-OPEN-03-51-3350 (V.N.T.) and the French National Research Agency grant NT05-241455GIPM. The third author would like to thank the Institute for Theoretical Physics, University of Wrocław for hospitality.

References

1. S. Doplicher, K. Fredenhagen, J.E. Roberts, Phys. Lett. B **331**, 39 (1994)
2. S. Doplicher, K. Fredenhagen, J.E. Roberts, Commun. Math. Phys. **172**, 187 (1995) [hep-th/0303037]
3. N. Seiberg, E. Witten, JHEP **9909**, 032 (1999) [hep-th/9908142]
4. R. Oeckl, Nucl. Phys. B **581**, 559 (2000) [hep-th/0003018]
5. J. Wess, Proc. of 2003 Workshop in Vrnacha Banya, Serbia, August 2003 (Belgrad, 2004), p. 122 [hep-th/0408080]
6. M. Chaichian, P.P. Kulish, K. Nishijima, A. Tureanu, Phys. Lett. B **604**, 98 (2004) [hep-th/0408069]
7. P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp, J. Wess, Class. Quantum Grav. **22**, 3511 (2005) [hep-th/0504183]
8. J. Lukierski, M. Woronowicz, Phys. Lett. B **633**, 116 (2006) [hep-th/0508083]
9. C. Gonera, P. Kosinski, P. Maslanka, S. Giller, Phys. Lett. B **622**, 192 (2005) [hep-th/0504132]
10. S. Zakrzewski, Lett. Math. Phys. **32**, 11 (1994)
11. S. Zakrzewski, Commun. Math. Phys. **187**, 285 (1997), http://arxiv.org/abs/q-al/9602001
12. A. Borowiec, J. Lukierski, V.N. Tolstoy, Czech. J. Phys. **55**, 11 (2005), http://xxx.lanl.gov/abs/hep-th/0301033
13. J. Lukierski, A. Nowicki, H. Ruegg, V.N. Tolstoy, Phys. Lett. B **264**, 331 (1991)
14. S.L. Woronowicz, S. Zakrzewski, Comput. Math. **90**, 211 (1994)
15. V.G. Drinfeld, in Proc. of XXth Int. Math. Congress (Berkeley, USA, 1986), p. 798
16. M. Jimbo, Lett. Math. Phys. **10**, 63 (1985)
17. W.B. Schmidke, J. Wess, B. Zumino, Z. Phys. C **52**, 471 (1991)
18. C. Ohn, Lett. Math. Phys. **25**, 85 (1992)
19. O.V. Ogievetsky, Suppl. Rend. Circ. Math. Palermo, Serie II **37**, 185 (1993), preprint MPI-Ph/92-99 (1992)
20. B. Abdesselam, A. Chakrabarti, R. Chakrabarti, J. Segar, http://arxiv.org/abs/q-alg/9807100
21. S. Majid, J. Math. Phys. **34**, 2045 (1993)
22. P. Podles, S.L. Woronowicz, Proc. of First Carribian Spring School of Mathematics and Theoretical Physics, June 1993, ed. by R. Coquereaux, M. Dubois-Violette, P. Flad, (Scientific, 1995), p. 364
23. V.G. Drinfeld, Leningrad Math. J. **1**, 1419 (1990)
24. P. Kulish, A. Mudrov, Proc. Steklov Inst. Math. **226**, 97 (1999), http://xxx.lanl.gov/abs/math.QA/9901019
25. A. Borowiec, J. Lukierski, V.N. Tolstoy, Eur. Phys. J. C **44**, 139 (2005) [hep-th/0412131]
26. V.N. Tolstoy, in Proc. of International Workshop "Supersymmetries and Quantum Symmetries (SQS'03)", Russia, Dubna, July, 2003, ed. by E. Ivanov, A. Pashnev (JINR, Dubna, 2004), p. 242, http://xxx.lanl.gov/abs/math.QA/0402433
27. P.P. Kulish, V.D. Lyakhovsky, A. Mudrov, J. Math. Phys. **24**, 4569 (1999)
28. L.D. Faddeev, N.Y. Reshetikhin, L.A. Takhtadjan, Algebra i Analiz **1**, 178 (1989)